8.4 – Matrices for General Linear Transformations

Recall this example from section 1.8

Find the standard matrix A for the linear transformation $T : R^2 \to R^3$ for which $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}3\\-5\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\1\end{bmatrix}$ and use it to compute $T\left(\begin{bmatrix}-4\\3\end{bmatrix}\right)$.

And #3 from 4.7

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \left\{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\right\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix}2\\1\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\-1\\1\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}1\\2\\1\end{bmatrix} \text{ and } \mathbf{u}_1' = \begin{bmatrix}3\\1\\-5\end{bmatrix}, \mathbf{u}_2' = \begin{bmatrix}1\\1\\-3\end{bmatrix}, \mathbf{u}_3' = \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

a. Find the transition matrix B to B'.

b. Compute the coordinate vector $[\mathbf{w}]_B$, where $\mathbf{w} = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$ and use the transition matrix in part (a) to compute $[\mathbf{w}]_{B'}$.

Now we combine these two concepts.

#5 Let $T : R^2 \to R^3$ be defined by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1 + 2x_2\\-x_1\\0\end{bmatrix}$.

a. Find the matrix $[T]_{B',B}$ relative to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix}1\\3\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2\\4\end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}3\\0\\0\end{bmatrix}$$

b. Verify that the formula $[T]_{B',B}[\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$ holds for every vector in R^2.

Let V be an n-dimensional vector space with basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and let W be an m-dimensional vector space with basis B'. Suppose that $T : V \to W$ is a linear transformation, and that for each vector $\mathbf{x}$ in V, the coordinate vectors for $\mathbf{x}$ and $T(\mathbf{x})$ are $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_{B'}$, respectively. Then the **matrix for T relative to the bases B and B′** is written [by this author] as
$[T]_{B', B} = \left[[T(\mathbf{u}_1)]_{B'} \middle| [T(\mathbf{u}_2)]_{B'} \middle| \dots \middle| [T(\mathbf{u}_n)]_{B'} \right]$.
This matrix has the property that $[T]_{B',B} [\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$.

#8 Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x-3)$, that is, $T\left(c_0 + c_1x + c_2x^2\right) = x\left(c_0 + c_1(x-3) + c_2(x-3)^2\right)$
a. Find $[T]_{B',B}$ relative to the bases $B = \left\{1,\ x,\ x^2\right\}$ and $B' = \left\{1,\ x,\ x^2, x^3\right\}$.
b. Use the three-step procedure illustrated in Example 2 to compute $T\left(1 + x + x^2\right)$.
c. Check the result obtained in part (b) by computing $T\left(1 + x - x^2\right)$ directly.

In the special case where $T : V \to V$ is a linear operator with $B = B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, the matrix for T is called the **matrix for T relative to the basis B** and is written [by this author] as $[T]_B = \left[[T(\mathbf{u}_1)]_B \,\middle|\, [T(\mathbf{u}_2)]_B \,\middle|\, \ldots \,\middle|\, [T(\mathbf{u}_n)]_B\right]$.

#18 Let $D : P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.
a. Find the matrix for D relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2 in which $\mathbf{p}_1 = 2$, $\mathbf{p}_2 = 2 - 3x$, $\mathbf{p}_3 = 2 - 3x + 8x^2$.
b. Use the matrix in part (a) to compute $D\left(6 - 6x + 24x^2\right)$.

Theorem 8.4.1 If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, and if B, B'', and B' are bases for U, V, and W, respectively, then
$[T_2 \circ T_2]_{B,\ B'} = [T_2]_{B',\ B''} [T_1]_{B'',\ B}$.

#12 Let $T_1 : P_1 \to P_2$ be the linear transformation defined by $T_1(p(x)) = xp(x)$ and let $T_2 : P_2 \to P_2$ be the linear operator defined by $T_2(p(x)) = p(2x+1)$. Let $B = \{1,\ x\}$ and $B' = \left\{1,\ x,\ x^2\right\}$ be the standard bases for P_1 and P_2.
a. Find $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B'}$, and $[T_1]_{B',B}$.
b. State a formula relating the matrices in part (a).
c. Verify that the matrices in part (a) satisfy the formula you stated in part (b).

Theorem 8.4.2

If $T : V \to V$ is a linear operator, and if B is a basis for V, then T is one-to-one if and only if $[T]_B$ is invertible. Moreover, when these conditions hold, $\left[T^{-1}\right]_B = [T]_B^{-1}$.